FORMS IN MANY VARIABLES

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1. Introduction

A system of homogeneous polynomials with rational coefficients has a non-trivial rational zero provided only that these polynomials are of odd degree, and the system has sufficiently many variables in terms of the number and degrees of these polynomials. While this striking theorem of Birch [1] addresses a fundamental diophantine problem in engagingly simple fashion, the problem of determining a satisfactory bound for the number of variables which suffice to guarantee the existence of a non-trivial zero remains unanswered in any but the simplest cases. Sophisticated versions of the Hardy-Littlewood method have been developed, first by Davenport [4] to show that 16 variables suffice for a single cubic form, and more recently by Schmidt [10] to show that $(10r)^5$ variables suffice for a system of r cubic forms. Unfortunately even Schmidt's highly developed version of the Hardy-Littlewood method is discouragingly ineffective in handling systems of higher degree (see [11, 12]). The object of this paper is to provide a method for obtaining explicit bounds for the number of variables required in Birch's Theorem. Our approach to this problem will involve the Hardy-Littlewood method only indirectly, being motivated by the elementary diagonalisation method of Birch. Although it has always been supposed that Birch's method would necessarily lead to bounds too large to be reasonably expressed, we are able to reconfigure the method so as to obtain estimates which in general are considerably sharper than those following from Schmidt's methods (see forthcoming work [15] for amplification of this remark). Indeed, for systems of quintic forms our new bounds might, at a stretch, be considered "reasonable".

In order to describe our conclusions we require some notation. When k is a field, d and r are natural numbers, and m is a non-negative integer, let $v_{d,r}^{(m)}(k)$ denote the least integer (if any such integer exists) with the property that whenever $s > v_{d,r}^{(m)}(k)$, and $f_i(\mathbf{x}) \in k[x_1, \dots, x_s]$ $(1 \le i \le r)$ are forms of degree d, then the system of equations $f_i(\mathbf{x}) = 0$ $(1 \le i \le r)$ possesses a solution set which contains a k-rational linear space of projective dimension m. If no such integer exists, define $v_{d,r}^{(m)}(k)$ to be $+\infty$. We abbreviate $v_{d,r}^{(0)}(k)$ to $v_{d,r}(k)$, and define $\phi_{d,r}(k)$ in like manner, save that the arbitrary forms of degree d are restricted to be diagonal.

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In view of the real solubility condition it is plain that $v_{d,r}^{(m)}(\mathbb{Q})$ can be finite only when d is odd. The simplest interesting examples to consider are therefore systems of cubic forms. In §3 we show how, for an arbitrary field k, one may bound $v_{3,r}^{(m)}(k)$ in terms of $\phi_{3,r}(k)$.

Theorem 1. *Let k be a field, let m and r be non-negative integers with $r \geq 1$, and suppose that $\phi_{3,r}(k)$ is finite. Then*

$$v_{3,r}^{(m)}(k) \leq r^3(m+1)^5\left(\phi_{3,r}(k)+1\right)^5.$$

We remark that a modification of our method, which we outline in §3 below, yields a bound of the shape

$$v_{3,r}^{(m)}(k) \ll (m+1)^{\alpha}, \tag{1.1}$$

where α is any number exceeding $\frac{1}{2}(5+\sqrt{17}) = 4.56155\dots$, and the implicit constant depends at most on k, r and α. Unfortunately the state of knowledge concerning upper bounds for $\phi_{d,r}(k)$ currently leaves much to be desired. Indeed, the only fields for which detailed investigations have thus far been executed are the local fields and $\mathbb{Q}$. Since we have considered local fields elsewhere (see [14]), we restrict attention to the case $k = \mathbb{Q}$, noting merely that recent developments in the theory of the Hardy-Littlewood method over algebraic number fields, when applied within Theorem 1, should yield useful bounds on $v_{3,r}^{(m)}(k)$ also when k is an algebraic field extension of $\mathbb{Q}$.

Corollary. *Let r be a natural number, and let m be a non-negative integer. Then*

$$v_{3,r}^{(m)}(\mathbb{Q}) < (90r)^8(\log(27r))^5(m+1)^5.$$

For comparison, Lewis and Schulze-Pillot [7, equation (4)] have provided an estimate of the shape

$$v_{3,r}^{(m)}(\mathbb{Q}) \ll r^{11}(m+1) + r^3(m+1)^5, \tag{1.2}$$

and have also indicated how to refine the latter bound for smaller m to obtain $v_{3,r}^{(m)}(\mathbb{Q}) \ll r^5(m+1)^{14}$. Thus the bound provided by the corollary to Theorem 1, which has strength

$$v_{3,r}^{(m)}(\mathbb{Q}) \ll_{\varepsilon} r^{8+\varepsilon}(m+1)^5,$$

is stronger than those of Lewis and Schulze-Pillot only for

$$r^{1/3+\varepsilon} \ll_{\varepsilon} m+1 \ll_{\varepsilon} r^{3/4-\varepsilon}.$$

Meanwhile, the improvement of our basic bound noted in (1.1) above yields a bound for $v_{3,r}^{(m)}(\mathbb{Q})$ superior to (1.2) whenever m is sufficiently large in terms of r (note, however, that a similar improvement may be put into effect in the work of Lewis and Schulze-Pillot). Also, when $r=1$, work of Wooley [13, Theorem 2(b)] shows that $v_{3,1}^{(m)}(\mathbb{Q}) \ll (m+1)^2$, whence $v_{3,2}^{(m)}(\mathbb{Q}) \ll (m+1)^4$. Of course, when $m=0$, so that one is seeking only the existence of rational points on the intersection of r cubic hypersurfaces, Schmidt's bound $v_{3,r}(\mathbb{Q}) < (10r)^5$ (see [10, Theorem 1]) is superior to the conclusion of the corollary.

In §4 we move on to the next most interesting class of examples, considering systems of quintic forms. Without any hypotheses concerning the behaviour of $\phi_{3,r}(k)$, unfortunately, the bounds on $v_{5,r}^{(m)}(k)$ stemming from our methods seem too complicated to merit mention. We therefore restrict attention to the rational field $\mathbb{Q}$. The sharpest estimates that we are able to derive for $v_{5,r}^{(m)}(\mathbb{Q})$ follow by exploiting an estimate of Lewis and Schulze-Pillot [7] for $v_{3,r}^{(m)}(\mathbb{Q})$ within the methods laid out in §2.

Theorem 2. *Let m and r be non-negative integers with $r \geq 1$. Then*

$$v_{5,r}^{(m)}(\mathbb{Q}) < \exp\left(10^{32}\left((m+1)r\log(3r)\right)^{\kappa}\log(3r(m+1))\right),$$

where

$$\kappa = \frac{\log 3430}{\log 4} = 5.87199\ldots.$$

In particular, $v_{5,r}(\mathbb{Q}) = o(e^{r^6})$.

For comparison, Schmidt [12, equation (2.5)] has shown that for a suitable positive constant A one has $v_{5,r}(\mathbb{Q}) \leq \exp(\exp(Ar))$. Thus our new result replaces a doubly exponential bound by one which is essentially single exponential.

We remark that when k is a field for which $\phi_i(k) < \infty$ for $2 \leq i \leq d$, as is the case, for example, for $\mathbb{Q}_p$ and its extensions, then Wooley [14, Theorem 2.4] has shown that

$$v_{d,r}^{(m)}(k) \leq 2(r^2\phi_d(k)+mr)^{2^{d-2}} \prod_{i=2}^{d-1}(\phi_i(k)+1)^{2^{i-2}}$$

(this sharpens an earlier result of Leep and Schmidt [6]; see also Schmidt [9] for a sharper conclusion for systems of cubic forms when $m=0$). Thus the assumption of a suitable local to global principle would lead, for odd d, to the bound

$$v_{d,r}^{(m)}(\mathbb{Q}) = \sup_{p \text{ prime}} v_{d,r}^{(m)}(\mathbb{Q}_p) \ll_d (r^2+mr)^{2^{d-2}},$$

a conclusion substantially stronger than those described in Theorems 1 and 2. In view of local obstructions, of course, one has the lower bound $v_{d,r}(\mathbb{Q}) \geq rd^2$, and some workers would even conjecture that the latter lower bound holds with equality for odd d.

For values of d larger than 5 our bound for $v_{d,r}(\mathbb{Q})$ is indescribably weaker, and our conclusions are considerably more complicated to explain. We will discuss such bounds, and the relevance of remarks of Schmidt [12, §2] in this context, on another occasion. Perhaps it is worth noting at this point, however, that subject to non-singularity conditions, stronger bounds are known for the number of variables required to solve systems of equations than have been derived herein (see Birch [2]). The point of the present paper, like that of Birch's original work [1], is to provide such conclusions without any hypotheses.

We describe our version of Birch's elementary diagonalisation argument in §2, this forming the core of our methods. In broad outline, our strategy is modelled on the original argument of Birch. Our superior conclusions stem from two sources. Firstly, by adapting an argument used by Lewis and Schulze-Pillot [7] to generate large dimensional linear spaces of rational solutions to systems of homogeneous cubic equations, we are able to efficiently generate large dimensional rational linear spaces on which a system of forms becomes diagonal. Roughly speaking, our argument doubles the dimension of the latter linear spaces with each iteration of the method, thereby leading to an exponential advantage over the methods available hitherto. Secondly, since we are able to apply this latter approach to a system of many forms simultaneously, we are able to exploit current knowledge concerning the solubility of systems of diagonal forms in order to avoid the inductive approach previously employed, in which large dimensional rational linear spaces of zeros of a single form are used to solve and remove one form at a time from the system. This second idea dramatically improves the quality of our conclusions.

Throughout, implicit constants in Vinogradov's notation $\ll$ and $\gg$ depend at most on the quantities occurring as subscripts to the notation.

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2. Reduction to diagonal forms

In this section we establish a reduction technique which, by rational change of variable, simplifies arbitrary systems of homogeneous polynomials into diagonal ones, albeit in far fewer variables. In order to fully implement our reduction argument we require some additional notation. Given an r-tuple of polynomials $\mathbf{F} = (F_1, \dots, F_r)$ with coefficients in a field k, denote by $\nu(\mathbf{F})$ the number of variables appearing explicitly in $\mathbf{F}$. We are interested in the existence of solutions, over k, of systems of homogeneous polynomial equations with coefficients in k. When such a solution set contains a linear subspace of the ambient space, we define its dimension to be that when considered as a projective space. When d is a positive odd integer, denote by $\mathcal{G}_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k)$ the set of $(r_d+r_{d-2}+\dots+r_1)$-tuples of homogeneous polynomials, of which r_i have degree i for $i = 1, 3, \dots, d$, with coefficients in k, which possess no non-trivial linear space of solutions of dimension m over k. We define $\mathcal{D}_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k)$ to be the corresponding set of diagonal homogeneous polynomials. We then define $w_d^{(m)}(\mathbf{r}) = w_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k)$ by

$$w_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k) = \sup_{\mathbf{g} \in \mathcal{G}_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k)} \nu(\mathbf{g}),$$

and we define $\phi_d^{(m)}(\mathbf{r}) = \phi_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k)$ by

$$\phi_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k) = \sup_{\mathbf{f} \in \mathcal{D}_d^{(m)}(r_d, r_{d-2}, \dots, r_1; k)} \nu(\mathbf{f}).$$

We observe for future reference that both $w_d^{(m)}(\mathbf{r})$ and $\phi_d^{(m)}(\mathbf{r})$ are increasing functions of the arguments m and r_i. For the sake of convenience we abbreviate $w_d^{(m)}(r, 0, \dots, 0; k)$ to $v_{d,r}^{(m)}(k)$, and note that $w_d^{(0)}(r, 0, \dots, 0; k) = v_{d,r}(k)$. We also abbreviate $\phi_d^{(m)}(r, 0, \dots, 0; k)$ to $\phi_{d,r}^{(m)}(k)$, and write $\phi_{d,r}(k)$ for $\phi_{d,r}^{(0)}(k)$.

Next, when $m \geq 2$, we define $\mathcal{H}_d^{(m)}(r; k)$ to be the set of r-tuples, $(F_1, \dots, F_r)$, of homogeneous polynomials of degree d, with coefficients in k, for which no linearly independent k-rational vectors $\mathbf{e}_1, \dots, \mathbf{e}_m$ exist such that $F_i(t_1\mathbf{e}_1 + \dots + t_m\mathbf{e}_m)$ is a diagonal form in $t_1, \dots, t_m$ for $1 \leq i \leq r$. We then define $\widetilde{w}_d^{(m)}(r) = \widetilde{w}_d^{(m)}(r; k)$ by

$$\widetilde{w}_d^{(m)}(r; k) = \sup_{\mathbf{h} \in \mathcal{H}_d^{(m)}(r)} \nu(\mathbf{h}).$$

Further, we adopt the convention that $\widetilde{w}_d^{(1)}(r; k) = 0$. Note that $\widetilde{w}_d^{(m)}(r; k)$ is an increasing function of the arguments m and r. Moreover, when $s > \widetilde{w}_d^{(m)}(r; k)$ and $F_1, \dots, F_r$ are homogeneous polynomials of degree d with coefficients in k possessing s variables, then there exist k-rational vectors $\mathbf{e}_1, \dots, \mathbf{e}_m$ with the property that $F_i(t_1\mathbf{e}_1 + \dots + t_m\mathbf{e}_m)$ is a diagonal form in $t_1, \dots, t_m$ for $1 \leq i \leq r$.

Lemma 2.1. *Let d be an odd integer with $d \geq 3$, and let r, n and m be natural numbers. Then*

$$\widetilde{w}_d^{(n+m)}(r; k) \leq s + w_{d-2}^{(M)}(\mathbf{R}; k),$$

where

$$M = \widetilde{w}_d^{(n)}(r; k), \quad s = 1 + w_{d-2}^{(N)}(\mathbf{S}; k), \quad N = \widetilde{w}_d^{(m)}(r; k),$$

and for $0 \leq u \leq (d-1)/2$,

$$R_{2u+1} = r\binom{s + d - 2u - 2}{d - 2u - 1} \quad \text{and} \quad S_{2u+1} = r\binom{n + d - 2u - 2}{d - 2u - 1}.$$

Proof. Write $\delta = (d-1)/2$, and take $\mathcal{N}$ to be an integer with $\mathcal{N} > s + w_{d-2}^{(M)}(\mathbf{R}; k)$. For $1 \leq j \leq r$, consider forms $\mathcal{F}_j$ of degree d, all having $\mathcal{N}$ variables. For $1 \leq j \leq r$ and $0 \leq u \leq \delta$ define the polynomials $\mathcal{G}_{ju}$ and $\mathcal{H}_{ju}$ through the expansion

$$\mathcal{F}_j(\mathbf{y} + t\mathbf{x}) = \sum_{u=0}^{\delta} \left(\mathcal{G}_{ju}(\mathbf{y}, \mathbf{x}) t^{2u+1} + \mathcal{H}_{ju}(\mathbf{y}, \mathbf{x}) t^{d-2u-1} \right), \tag{2.1}$$

valid for each $\mathbf{x},\mathbf{y}\in k^{\mathcal{N}}$. Notice that $\mathcal{G}_{ju}(\mathbf{y},\mathbf{x})$ is a form of degree $2u+1$ in $\mathbf{x}$, and of degree $d-2u-1$ in $\mathbf{y}$. Also, $\mathcal{H}_{ju}(\mathbf{y},\mathbf{x})$ is a form of degree $d-2u-1$ in $\mathbf{x}$, and of degree $2u+1$ in $\mathbf{y}$. Let T be an arbitrary, but fixed, k-linear subspace of $k^{\mathcal{N}}$ of affine dimension s, and let $\mathbf{a}_1,\dots,\mathbf{a}_s$ be a basis for T. Let U be any subspace of $k^{\mathcal{N}}$ such that $T\oplus U=k^{\mathcal{N}}$. Consider an arbitrary element of T, say $\mathbf{y}=u_1\mathbf{a}_1+\dots+u_s\mathbf{a}_s$, and substitute this expression into $\mathcal{G}_{ju}(\mathbf{y},\mathbf{x})$. We find that the latter polynomial becomes a form of degree $d-2u-1$ in $u_1,\dots,u_s$, whose coefficients are forms of degree $2u+1$ in $\mathbf{x}$. Moreover, following a simple counting argument, one finds that the number of such coefficients of degree $2u+1$ is

$$\binom{(d-2u-1)+(s-1)}{d-2u-1}.$$

Thus, as we consider all $\mathcal{G}_{ju}(\mathbf{y},\mathbf{x})$ with $1\le j\le r$, we find that the total number of coefficients of degree $2u+1$ which arise is R_{2u+1} $(0\le u\le \delta)$. Then since $\mathcal{N}-s>w_{d-2}^{(M)}(\mathbf{R};k)$, we may conclude thus far that there exists a k-linear subspace, V, of U, with projective dimension M, on which all of the above coefficients of degrees $1,3,\dots,d-2$ vanish. Moreover, for each j one has $\mathcal{G}_{j\delta}(\mathbf{y},\mathbf{x})=\mathcal{F}_j(\mathbf{x})$. Consequently, for each $\mathbf{x}\in V$ and each $\mathbf{y}\in T$ one has

$$\mathcal{F}_j(\mathbf{y}+t\mathbf{x})=t^d\mathcal{F}_j(\mathbf{x})+\sum_{u=0}^{\delta}\mathcal{H}_{ju}(\mathbf{y},\mathbf{x})t^{d-2u-1}\quad (1\le j\le r). \tag{2.2}$$

Next, since $M+1>\widetilde{w}_d^{(n)}(r;k)$, we deduce that there exist linearly independent vectors $\mathbf{b}_1,\dots,\mathbf{b}_n\in V$ with the property that for each $t_1,\dots,t_n\in k$ one has for $1\le j\le r$ that $\mathcal{F}_j(t_1\mathbf{b}_1+\dots+t_n\mathbf{b}_n)$ is a diagonal form in $t_1,\dots,t_n$. Let W be the linear subspace of $k^{\mathcal{N}}$ spanned by $\mathbf{b}_1,\dots,\mathbf{b}_n$, and consider an arbitrary element of W, say $\mathbf{x}=v_1\mathbf{b}_1+\dots+v_n\mathbf{b}_n$. On substituting the latter expression into $\mathcal{H}_{ju}(\mathbf{y},\mathbf{x})$, we find that the latter polynomial becomes a form of degree $d-2u-1$ in $v_1,\dots,v_n$, whose coefficients are forms of degree $2u+1$ in $\mathbf{y}$. Moreover, following a simple counting argument, one finds that the number of such coefficients of degree $2u+1$ is

$$\binom{(d-2u-1)+(n-1)}{d-2u-1}.$$

Thus, as we consider all $\mathcal{H}_{ju}(\mathbf{y},\mathbf{x})$ with $1\le j\le r$, we find that the number of coefficients of degree $2u+1$ which arise is S_{2u+1} $(0\le u\le\delta)$. Then since $s>w_{d-2}^{(N)}(\mathbf{S};k)$, we may conclude that there exists a k-linear subspace, X, of T, with projective dimension N, on which all of the above coefficients of degrees $1,3,\dots,d-2$ vanish. Moreover, for each j one has $\mathcal{H}_{j\delta}(\mathbf{y},\mathbf{x})=\mathcal{F}_j(\mathbf{y})$. Consequently, for each $\mathbf{y}\in X$ and each $\mathbf{x}\in W$ one has

$$\mathcal{F}_j(\mathbf{y}+t\mathbf{x})=t^d\mathcal{F}_j(\mathbf{x})+\mathcal{F}_j(\mathbf{y})\quad (1\le j\le r). \tag{2.3}$$

Thus, since the affine dimension of X is $N+1$ and $N+1>\widetilde{w}_d^{(m)}(r;k)$, we deduce that there exist linearly independent vectors $\mathbf{c}_1,\dots,\mathbf{c}_m\in X$ with the property that for each $s_1, \dots, s_m \in k$ one has for $1 \le j \le r$ that $\mathcal{F}_j(s_1\mathbf{c}_1 + \cdots + s_m\mathbf{c}_m)$ is a diagonal form in $s_1, \dots, s_m$. Consequently, when $1 \le j \le r$,

$$\mathcal{F}_j(t_1\mathbf{b}_1 + \cdots + t_n\mathbf{b}_n + s_1\mathbf{c}_1 + \cdots + s_m\mathbf{c}_m)$$

is a diagonal form in $t_1, \dots, t_n, s_1, \dots, s_m$, whence $\widetilde{w}_d^{(n+m)}(r;k) < \mathcal{N}$, and the lemma follows immediately.

Recalling the trivial result $\widetilde{w}_d^{(1)}(r;k) = 0$, it is apparent that Lemma 2.1 may be exploited inductively to obtain bounds for $\widetilde{w}_d^{(m)}(r;k)$ for arbitrary m. We now indicate how to bound $v_{d,r}^{(m)}(k)$ in terms of $\widetilde{w}_d^{(n)}(r;k)$ for suitable n.

Lemma 2.2. *Let d be an odd positive number, let r be a natural number, and let m be a non-negative integer. Then*

$$v_{d,r}^{(m)}(k) \le \widetilde{w}_d^{(M)}(r;k),$$

where $M = (m+1)(\phi_{d,r}(k) + 1)$.

Proof. Take N to be an integer with $N > \widetilde{w}_d^{(M)}(r;k)$, and for $1 \le j \le r$, consider forms $\mathcal{F}_j$ of degree d, all having N variables. By the definition of $\widetilde{w}_d^{(M)}(r;k)$, there exist linearly independent k-rational vectors $\mathbf{e}_1, \dots, \mathbf{e}_M$ with the property that whenever $t_1, \dots, t_M \in k$, one has for $1 \le j \le r$ that the form $\mathcal{F}_j(t_1\mathbf{e}_1 + \cdots + t_M\mathbf{e}_M)$ is a diagonal form in $t_1, \dots, t_M$. Let c_{ij} $(1 \le i \le r,\ 1 \le j \le M)$ be elements of k such that

$$\mathcal{F}_i(t_1\mathbf{e}_1 + \cdots + t_M\mathbf{e}_M) = \sum_{j=1}^{M} c_{ij} t_j^d \quad (1 \le i \le r). \tag{2.4}$$

Write $\phi = 1 + \phi_{d,r}(k)$. We observe that, by the definition of $\phi_{d,r}(k)$, for $l = 0, 1, \dots, m$, each of the systems of equations

$$\sum_{j=l\phi+1}^{(l+1)\phi} c_{ij} t_j^d = 0 \quad (1 \le i \le r)$$

possesses a non trivial k-rational solution. Consequently, there exist linearly independent k-rational vectors $\mathbf{a}_0, \dots, \mathbf{a}_m$ such that for each $u_0, \dots, u_m \in k$ one has

$$\mathcal{F}_j(u_0\mathbf{a}_0 + \cdots + u_m\mathbf{a}_m) = 0 \quad (1 \le j \le r).$$

Thus the system of equations $\mathcal{F}_j(\mathbf{x}) = 0$ $(1 \le j \le r)$ possesses a linear space of solutions of projective dimension m, whence $v_{d,r}^{(m)}(k) \le N$. This completes the proof of the lemma.

It is now clear how to bound $v_{d,r}^{(m)}(k)$ in terms of $w_{d-2}^{(n)}(\mathbf{s};k)$, for suitable n and $\mathbf{s}$ depending on d, m and r, provided of course that we have sufficient knowledge concerning the solubility of systems of diagonal equations. We turn to the latter issue in §3. Our next lemma completes the preliminaries necessary to facilitate our induction by bounding $w_{d-2}^{(n)}(s_{d-2}, \dots, s_1;k)$ in terms of $v_{d-2,r}^{(n)}(k)$ and $w_{d-4}^{(p)}(\mathbf{t};k)$ for suitable r, p and $\mathbf{t}$.

Lemma 2.3. *Let d be an odd positive number with $d \geq 3$, and let $r_1, r_3, \ldots, r_d$ be non-negative integers with $r_d > 0$. Then for each non-negative integer m one has*

$$w_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k) \leq w_{d-2}^{(M)}(r_{d-2}, \ldots, r_1; k),$$

where $M = v_{d,r_d}^{(m)}(k)$.

Proof. Take N to be an integer with $N > w_{d-2}^{(M)}(r_{d-2}, \ldots, r_1; k)$. Consider forms $\mathcal{F}_{ij}$ of degree i for $1 \leq j \leq r_i$ and $i = 1, 3, \ldots, d$, all having N variables and coefficients in k. By the definition of $w_{d-2}^{(M)}(\mathbf{r}; k)$, there exists a k-linear solution set of the system of equations

$$\mathcal{F}_{ij}(\mathbf{x}) = 0 \quad (1 \leq j \leq r_i,\ i = 1, 3, \ldots, d-2)$$

with projective dimension M. Let $\mathbf{e}_0, \ldots, \mathbf{e}_M$ be a basis for the latter space of solutions. Then for each $t_0, \ldots, t_M \in k$ one has

$$\mathcal{F}_{ij}(t_0\mathbf{e}_0 + \cdots + t_M\mathbf{e}_M) = 0 \quad (1 \leq j \leq r_i,\ i = 1, 3, \ldots, d-2).$$

Moreover, for $1 \leq j \leq r_d$, each of the forms $\mathcal{F}_{dj}(t_0\mathbf{e}_0 + \cdots + t_M\mathbf{e}_M)$ is a form of degree d in the $M + 1$ variables $t_0, \ldots, t_M$. Thus, since $M + 1 > v_{d,r_d}^{(m)}(k)$, there exists a k-linear solution set of the system of equations

$$\mathcal{F}_{dj}(t_0\mathbf{e}_0 + \cdots + t_M\mathbf{e}_M) = 0 \quad (1 \leq j \leq r_d)$$

with projective dimension m. Let $\mathbf{a}_0, \ldots, \mathbf{a}_m$ be a basis for the latter space of solutions. Then for each $u_0, \ldots, u_m \in k$ one has

$$\mathcal{F}_{ij}(u_0\mathbf{a}_0 + \cdots + u_m\mathbf{a}_m) = 0 \quad (1 \leq j \leq r_i,\ i = 1, 3, \ldots, d),$$

whence $w_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k) < N$. This completes the proof of the lemma.

3. Systems of cubic forms

Before embarking on our primary course, we detour in this section to discuss the existence of rational linear spaces in the solution set of systems of homogeneous cubic equations. This topic has been addressed in considerable generality by Lewis and Schulze-Pillot (see [7]), and more recently for a single equation in [13]. The conclusions of Lewis and Schulze-Pillot rest on the deep work of Schmidt [10]. Our methods, although elementary, yield superior conclusions to the aforementioned results whenever the dimension of the linear space lies in an interval intermediate in size in terms of the number of forms. We observe also that our methods apply in any field k for which suitable upper bounds are available for $\phi_{3,r}(k)$.

The proof of Theorem 1. We start by using Lemma 2.1 to bound $\widetilde{w}_3^{(n)}(r; k)$ as a function of n. Recall the notation of the statement of Lemma 2.1. Take $d = 3$, so that $R_1 = r\binom{s+1}{2}$ and $S_1 = r\binom{n+1}{2}$. We also take $m = 1$, so that $N = \widetilde{w}_3^{(1)}(r;k) = 0$ and

$$s = 1 + w_1^{(N)}(S_1;k) = 1 + S_1 = 1 + r\binom{n+1}{2}.$$

Notice that when $n \geq 2$, one has $s \leq rn^2$. Further,

$$w_1^{(M)}(R_1;k) = R_1 + M = r\binom{s+1}{2} + \widetilde{w}_3^{(n)}(r;k).$$

On inserting these estimates into Lemma 2.1, we find that

$$\widetilde{w}_3^{(n+1)}(r;k) \leq s + r\binom{s+1}{2} + \widetilde{w}_3^{(n)}(r;k), \tag{3.1}$$

and hence when $n \geq 2$ that

$$\begin{aligned}\widetilde{w}_3^{(n+1)}(r;k) &\leq \widetilde{w}_3^{(n)}(r;k) + rn^2 + \tfrac{1}{2}r^2n^2(rn^2+1)\\ &\leq \widetilde{w}_3^{(n)}(r;k) + r^3n^4.\end{aligned} \tag{3.2}$$

Moreover, $\widetilde{w}_3^{(1)}(r;k) = 0$, and hence by (3.1), on noting that when $n = 1$ one has $s = r + 1$, we deduce that

$$\widetilde{w}_3^{(2)}(r;k) \leq 1 + r + r\binom{r+2}{2} + \widetilde{w}_3^{(1)}(r;k) < 5r^3 < 2^5r^3. \tag{3.3}$$

On applying (3.2), we therefore deduce that when $n \geq 2$,

$$\widetilde{w}_3^{(n+1)}(r;k) < r^3\left(5 + \sum_{m=2}^{n} m^4\right) < n^5r^3. \tag{3.4}$$

On recalling (3.3), we conclude that $\widetilde{w}_3^{(n)}(r;k) < r^3n^5$ for each positive integer n.

Finally we apply Lemma 2.2, so that by (3.4) we arrive at the estimate

$$v_{3,r}^{(m)}(k) \leq r^3(m+1)^5\left(\phi_{3,r}(k)+1\right)^5.$$

This completes the proof of the theorem.

By altering the choice of m in the above argument we obtain the bound (1.1) discussed in the introduction.

Theorem 3.1. *Let k be a field, let m and r be non-negative integers with $r \geq 1$, and suppose that $\phi_{3,r}(k)$ is finite. Then whenever $\alpha > \frac{1}{2}(5+\sqrt{17})$, one has*

$$v_{3,r}^{(m)}(k) \ll_{k,r,\alpha} (m+1)^{\alpha}.$$

Proof. We form the hypothesis that for some positive number β, with $\beta > \frac{1}{2}(5+\sqrt{17})$, one has

$$\widetilde{w}_3^{(n)}(r;k) \ll n^{\beta}, \tag{3.5}$$

where here, and throughout the rest of the proof of this theorem, the implicit constant depends at most on k, r and β. We mimic the argument of the proof of Theorem 1, but now take $m = [n^{2/\beta}] + 1$. Thus, in the notation of the statement of Lemma 2.1, we have

$$s = 1 + w_1^{(N)}(S_1;k) = 1 + S_1 + \widetilde{w}_3^{(m)}(r;k),$$

whence by (3.5) one obtains

$$s \ll n^2 + m^{\beta} \ll n^2.$$

Next we recall that

$$w_1^{(M)}(R_1;k) = r\binom{s+1}{2} + \widetilde{w}_3^{(n)}(r;k),$$

and thus deduce from Lemma 2.1 that

$$\widetilde{w}_3^{(n+m)}(r;k) - \widetilde{w}_3^{(n)}(r;k) \ll s^2 \ll n^4.$$

A trivial induction now reveals that for each positive integer n,

$$\widetilde{w}_3^{(n)}(r;k) \ll n^5/m \ll n^{5-2/\beta},$$

whence the hypothesis (3.5) holds with β replaced by $5 - 2/\beta$. In view of (3.4), we therefore conclude that the hypothesis (3.5) holds with β replaced by the exponent β_r, for any $r \in \mathbb{N}$, where β_r is defined by $\beta_1 = 5$, and $\beta_{r+1} = 5 - 2/\beta_r$ $(r \in \mathbb{N})$. After verifying that $\lim_{r\to\infty} \beta_r = \frac{1}{2}(5+\sqrt{17})$, the proof of the theorem is complete.

In order to establish the corollary to Theorem 1 we will require an estimate for $\phi_{3,r}(\mathbb{Q})$. We record for this and future use the following lemma.

Lemma 3.2. *Let d and r be natural numbers with d odd. Then*

$$\phi_{d,r}(\mathbb{Q}) + 1 \le 48rd^3 \log\left(3rd^2\right).$$

Proof. This is immediate from the corollary to Theorem 1 of Brüdern and Cook [3], the latter making fundamental use of the corresponding local results of Low, Pitman and Wolff [8].

We note that older results of Davenport and Lewis [5] would also yield reasonable, though somewhat weaker, conclusions when exploited within our methods.

We are now in a position to prove the corollary to Theorem 1, which provides an estimate for the number of variables required to guarantee the existence of a rational m-dimensional linear space of solutions on the intersection of a number of cubic hypersurfaces.

The proof of the corollary to Theorem 1. We apply Theorem 1, bounding $\phi_{3,r}(\mathbb{Q})$ by using Lemma 3.2. Thus

$$\phi_{3,r}(\mathbb{Q}) + 1 < 6^4 r \log(27r),$$

and hence

$$v_{3,r}^{(m)}(\mathbb{Q}) \leq 6^{20} r^8 (\log(27r))^5 (m+1)^5,$$

and the corollary follows immediately.

Since it is useful for our discussion, in the following section, of systems of quintic forms, we record an explicit version of the bound of Lewis and Schulze-Pillot.

Lemma 3.3. *Let r be a natural number, and let m be a non-negative integer. Then*

$$v_{3,r}^{(m)}(\mathbb{Q}) < (11r)^{11}(m+1) + 50r^3(m+1)^5.$$

Proof. We employ the bounds on $v_{3,r}^{(m)}(\mathbb{Q})$ used by Lewis and Schulze-Pillot [7] in their proof of [7, inequality (4)], being careful to keep all intermediate estimates explicit. In combination with Schmidt's bound $v_{3,r}^{(0)}(\mathbb{Q}) < (10r)^5$ (see [10, Theorem 1]), the argument of Lewis and Schulze-Pillot yields

$$\begin{aligned} v_{3,r}^{(m)}(\mathbb{Q}) &\leq v_{3,r}^{(0)}(\mathbb{Q}) + 3r \sum_{j=1}^{m} \left(v_{3,r}^{(0)}(\mathbb{Q}) + 3rj^2 + 2 \right)^2 \\ &< (10r)^5 + 3r \sum_{j=1}^{m} \left((10r)^5 + 3rj^2 + 2 \right)^2, \end{aligned}$$

and the desired conclusion follows with a modicum of computation.

4. Systems of quintic forms

We now return to our major goal, that of bounding $v_{5,r}^{(m)}(\mathbb{Q})$. Once again the key to our argument is Lemma 2.1, and again we make use of the estimate for $\phi_{d,r}(\mathbb{Q})$ provided by Lemma 3.2. We begin with a lemma which bounds $\widetilde{w}_5^{(n)}(r;\mathbb{Q})$ as a function of n and r. The conclusion of the lemma represents a compromise between strength and simplicity. We remark on some possible improvements at the end of this section.

Lemma 4.1. *For each non-negative integer j one has*

$$\widetilde{w}_5^{(m_j)}(r;\mathbb{Q}) < (2rm_j^2)^{3\alpha_j}, \tag{4.1}$$

where

$$m_j = 4^j \quad \text{and} \quad \alpha_j = 3430^j.$$

Proof. We use induction to establish that for each j one has

$$\widetilde{w}_5^{(m_j)}(r;\mathbb{Q}) < e^{\beta_j} r^{\gamma_j} m_j^{\delta_j}, \tag{4.2}$$

where $m_j = 4^j$, and

$$\beta_j = 3982 \cdot \frac{3430^j - 1}{3429}, \quad \gamma_j = 9261 \cdot \frac{3430^j - 1}{3429}, \quad \delta_j = 17836 \cdot \frac{3430^j - 1}{3429}. \tag{4.3}$$

Note first that (4.2) holds trivially when $j = 0$. We suppose next that (4.2) holds for a non-negative integer j, and aim to establish that (4.2) holds with j replaced by $j+1$. Let r, n and m be natural numbers. Recall the notation of the statement of Lemma 2.1, and take $d = 5$. Then

$$S_1 = r\binom{n+3}{4} \le rn^4 \quad \text{and} \quad S_3 = r\binom{n+1}{2} \le rn^2,$$

whence by Lemma 3.3, on writing $N = \widetilde{w}_5^{(m)}(r;\mathbb{Q})$ and $N_1 = N + 1$, one has that

$$\begin{aligned} s = 1 + w_3^{(N)}(S_3, S_1; \mathbb{Q}) &= 1 + S_1 + v_{3,S_3}^{(N)}(\mathbb{Q}) \\ &< 1 + rn^4 + (11rn^2)^{11} N_1 + 50(rn^2)^3 N_1^5 \\ &< C_1 (rn^2)^3 (N_1^5 + (rn^2)^{10}), \end{aligned} \tag{4.4}$$

where

$$C_1 = 11^{11} + 50 + 2 < e^{27}. \tag{4.5}$$

Next, on writing $M = \widetilde{w}_5^{(n)}(r;\mathbb{Q})$ and $M_1 = M + 1$, one has from Lemma 2.1,

$$\begin{aligned} \widetilde{w}_5^{(n+m)}(r;\mathbb{Q}) &\le s + w_3^{(M)}(R_3, R_1; \mathbb{Q}) = s + R_1 + v_{3,R_3}^{(M)}(\mathbb{Q}) \\ &< s + rs^4 + (11rs^2)^{11} M_1 + 50(rs^2)^3 M_1^5 \\ &< C_1 (rs^2)^3 (M_1^5 + (rs^2)^{10}). \end{aligned} \tag{4.6}$$

On substituting from (4.4) into (4.6), we find that

$$\begin{aligned} \widetilde{w}_5^{(n+m)}(r;\mathbb{Q}) &< C_1^7 r^{21} n^{36} \left(N_1^5 + (rn^2)^{10}\right)^6 \left(M_1^5 + C_1^{20} r^{70} n^{120} \left(N_1^5 + (rn^2)^{10}\right)^{20}\right) \\ &< C_2 \left(r^{351} n^{676} + r^{91} n^{156} N_1^{130} + r^{21} n^{36} M_1^5 N_1^{30} + r^{81} n^{156} M_1^5\right), \end{aligned} \tag{4.7}$$

where
$$C_2 = 2^6 C_1^7 (1 + (2C_1)^{20}) < e^{748}. \tag{4.8}$$

First we take $n = m$ in (4.7) to obtain

$$\widetilde{w}_5^{(2m)}(r;\mathbb{Q}) < 4C_2 \left(r^{351} m^{676} + r^{91} m^{156} N_1^{130} \right), \tag{4.9}$$

where we recall that $N_1 = 1 + \widetilde{w}_5^{(m)}(r;\mathbb{Q})$. Next, on taking $n = 2m$ and making use of (4.9) in (4.7), we deduce that

$$\widetilde{w}_5^{(3m)}(r;\mathbb{Q}) < C_3 \left(r^{1836} m^{3536} + r^{536} m^{936} N_1^{680} \right), \tag{4.10}$$

where
$$C_3 = C_2 (1 + 8C_2)^5 \left(2^{676} + 2^{157} + 2^{36} \right) < e^{4967}. \tag{4.11}$$

Finally, on taking $n = 3m$ and making use of (4.10) in (4.7), we conclude that

$$\widetilde{w}_5^{(4m)}(r;\mathbb{Q}) < C_4 \left(r^{9261} m^{17836} + r^{2761} m^{4836} N_1^{3430} \right), \tag{4.12}$$

where
$$C_4 = C_2 (1 + 2C_3)^5 \left(3^{676} + 3^{157} + 3^{36} \right) < e^{26330}. \tag{4.13}$$

Now recall the inductive hypothesis (4.2). We deduce from (4.12) and (4.13) that

$$\begin{aligned} \widetilde{w}_5^{(4m_j)}(r;\mathbb{Q}) &< C_4 \left(r^{9261} m_j^{17836} + r^{2761} m_j^{4836} \left(1 + e^{\beta_j} r^{\gamma_j} m_j^{\delta_j} \right)^{3430} \right) \\ &< C_4 (1 + 2^{3430}) e^{3430\beta_j} r^{9261 + 3430\gamma_j} m_j^{17836 + 3430\delta_j} \\ &< e^{3982 + 3430\beta_j} r^{9261 + 3430\gamma_j} m_{j+1}^{17836 + 3430\delta_j}. \end{aligned}$$

Then in view of (4.3) we deduce that

$$\widetilde{w}_5^{(m_{j+1})}(r;\mathbb{Q}) < e^{\beta_{j+1}} r^{\gamma_{j+1}} m_{j+1}^{\delta_{j+1}},$$

whence the inductive hypothesis follows with j replaced by $j+1$. We may therefore conclude that (4.2) holds for all non-negative integers j. Finally, (4.1) follows from (4.2) with a little calculation.

It is now possible to bound $v_{5,r}^{(m)}(\mathbb{Q})$ by combining the bound for $\phi_{5,r}(\mathbb{Q})$ provided by Lemma 3.2 together with Lemma 2.2.

The proof of Theorem 2. By Lemma 2.2 one has

$$v_{5,r}^{(m)}(\mathbb{Q}) \le \widetilde{w}_5^{(M)}(r;\mathbb{Q}),$$

where $M=(m+1)(\phi_{5,r}(\mathbb{Q})+1)$. We may therefore apply Lemma 4.1 with

$$j=\left[\frac{\log M}{\log 4}\right]+1$$

to obtain, in the notation of the statement of Lemma 4.1,

$$v_{5,r}^{(m)}(\mathbb{Q})\le \widetilde{w}_5^{(m_j)}(r;\mathbb{Q})<(2rm_j^2)^{3\alpha_j},$$

where

$$\alpha_j=3430^j<\exp\left(\left(\frac{\log M}{\log 4}+1\right)\log 3430\right)<3430M^{\kappa},$$

where $\kappa=(\log 3430)/(\log 4)$. Thus we deduce that

$$\log v_{5,r}^{(m)}(\mathbb{Q})<10290M^{\kappa}\log(32rM^2).$$

But by Lemma 3.2 one has

$$\phi_{5,r}(\mathbb{Q})+1\le 6000r\log(75r),$$

whence, following a modicum of computation, one deduces that

$$\log v_{5,r}^{(m)}(\mathbb{Q})<10^{32}\left((m+1)r\log(3r)\right)^{\kappa}\log(3r(m+1)).$$

This completes the proof of the theorem.

If, in the proof of Lemma 4.1, the conclusion of the corollary to Theorem 1 is applied in place of Lemma 3.3 to obtain a substitute for (4.6), one arrives at an expression of the shape

$$\widetilde{w}_5^{(n+m)}(r;\mathbb{Q})\ll_{\varepsilon}(rs^2)^{8+\varepsilon}M_1^5,$$

whence from (4.4) one deduces that

$$\widetilde{w}_5^{(2m)}(r;\mathbb{Q})\ll_{\varepsilon}(rm)^{O(1)}\left(1+\widetilde{w}_5^{(m)}(r;\mathbb{Q})\right)^{85+\varepsilon}.$$

On taking $n=2m$ and making use of the latter bound in (4.7), one obtains

$$\widetilde{w}_5^{(3m)}(r;\mathbb{Q})\ll_{\varepsilon}(rm)^{O(1)}\left(1+\widetilde{w}_5^{(m)}(r;\mathbb{Q})\right)^{455+\varepsilon}.$$

Thus a theorem of similar shape to Theorem 2 may be established, save that κ is now replaced by any number exceeding

$$\frac{\log 455}{\log 3}=5.57093\dots.$$

A further modest reduction in the permissible value of κ would be made possible by a suitable version of the bound $v_{3,r}^{(m)}(\mathbb{Q})\ll r^5(m+1)^{14}$ claimed by Lewis and Schulze-Pillot [7].

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